plate dimensions; n, number of elements on the plate; $1\{U_i\}$, trace using a step function of a discrete heat source; τ, time; $t_i(\tau)$, $\vartheta_i(\tau)$, temperature and overheating of the plate at the point of attachment of the j-th element at time τ; m, rate of heating of the plate; C, total heat capacity of the structure (plate + elements); ψ, criterion for uniformity of the temperature field, equal to the ratio of the average surface temperature of the plate to its average volume temperature; σ, thermal conductivity from the plate to the medium; σ_m, σ_p, thermal conductivities from the housing of the element to the medium and into the plate; ϑ_j, average surface overheating of the element housing in the nonstationary state; m_0, rate of heating of the element; R_{in}, internal thermal resistance between the crystal and the element housing.

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USE OF CLASSICAL STEFAN PROBLEM FOR INITIALIZING SOLUTION IN THE NUMERICAL PREDICTION OF FREEZING

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The exact solution of the classical Stefan problem is examined from the point of view of using it as an "initial solution" in numerical solutions of appropriate problems.

The classical Stefan problem is taken to mean the self-similar one-dimensional problem of freezing or melting of a homogeneous isotropic medium with constant boundary conditions [1].

The solution of such a problem can be represented in the form

$$T_u(z,\ t) = T_w + \frac{T_f - T_w}{\operatorname{erf}\left(\dfrac{\beta}{2\sqrt{a_u}}\right)} \operatorname{erf}\left(\frac{z}{2\sqrt{a_u t}}\right), \tag{1}$$

$$T_v(z,\ t) = T_0 - \frac{T_0 - T_f}{\operatorname{erfc}\left(\dfrac{\beta}{2\sqrt{a_v}}\right)} \operatorname{erfc}\left(\frac{z}{2\sqrt{a_v t}}\right), \tag{2}$$

$$\zeta(t) = \beta\sqrt{t}. \tag{3}$$

The coefficient of proportionality β, characterizing the velocity of the phase transition

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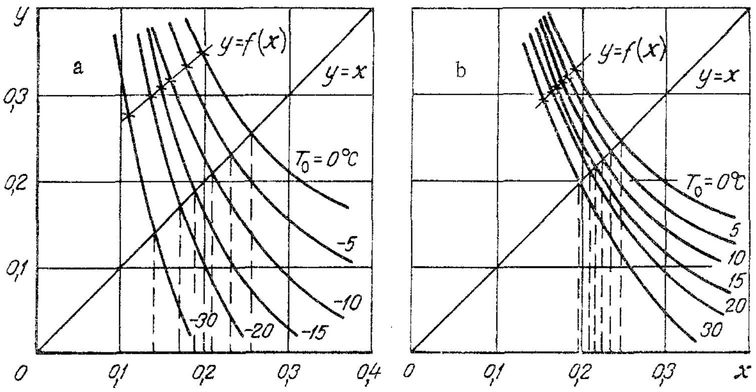

Fig. 1. Finding the root of Eqs. (4') and (5'):
a) T_w = +10°C; b) T_w = −20°C.

front (PTF), is found from a transcendental equation, which can be represented in the following form [2, 3]:

for freezing

$$\frac{\lambda_1 (T_f - T_w) \exp\left(-\frac{\beta^2}{4a_1}\right)}{\sqrt{a_1}\,\mathrm{erf}\left(\frac{\beta}{2\sqrt{a_1}}\right)} - \frac{\lambda_2 (T_0 - T_f) \exp\left(-\frac{\beta^2}{4a_2}\right)}{\sqrt{a_2}\,\mathrm{erfc}\left(\frac{\beta}{2\sqrt{a_2}}\right)} = \frac{L\gamma_2\sqrt{\pi}}{2}\beta, \tag{4}$$

for melting

$$\frac{\lambda_2 (T_w - T_f) \exp\left(-\frac{\beta^2}{4a_2}\right)}{\sqrt{a_2}\,\mathrm{erf}\left(\frac{\beta}{2\sqrt{a_2}}\right)} - \frac{\lambda_1 (T_f - T_0) \exp\left(-\frac{\beta^2}{4a_1}\right)}{\sqrt{a_1}\,\mathrm{erfc}\left(\frac{\beta}{2\sqrt{a_1}}\right)} = \frac{L\gamma_1\sqrt{\pi}}{2}\beta. \tag{5}$$

The existence of a positive root of Eqs. (4) and (5) follows from the fact that as β varies from zero to $+\infty$ the left side of these equations varies continuously from $+\infty$ to $-\infty$. The uniqueness of the root follows from the fact that the left sides of these equations are monotonic functions of β, while the right sides represent a straight line.

In the final analysis, the solution of Stefan's problem reduces to finding the root of Eq. (4) in the case of freezing and Eq. (5) in the case of melting.

The practical application of the exact solution presented is very restricted. With arbitrary changes in the formulation, compared to the self-similar problem, as well as in the multidimensional case, it is only possible to obtain a numerical solution [4-6].

In its turn, in the numerical solution of Stefan-type problems using the grid method with explicit separation of the moveable interface separating the phases, it is necessary to know the location of the PTF at a time taken as the initial time and the corresponding temperature distribution in both phases. For this purpose the exact solution (1-3) presented above turns out to be extremely useful as an "initial solution."

Taking into account the fact that the process of calculating the left sides of Eqs. (4) and (5) presents certain difficulties and that a series of calculations is usually performed in a numerical solution, it makes sense to solve Eqs. (4) or (5) using a computer.

For convenience in the calculation and for generalizing the results obtained on a computer, we shall introduce the dimensionless variables

$$x_1 = \beta/(2\sqrt{a_1}), \quad x_2 = \beta/(2\sqrt{a_2}), \tag{6}$$

$$A_{\mathrm{fr}} = \frac{C_1 (T_f - T_w)}{L\sqrt{\pi}}\,\frac{\gamma_1}{\gamma_2}, \quad B_{\mathrm{fr}} = \frac{C_2 (T_0 - T_f)}{L\sqrt{\pi}}\,\frac{\sqrt{a_2}}{\sqrt{a_1}}, \tag{7}$$

$$A_{\mathrm{melt}} = \frac{C_2 (T_w - T_f)}{L\sqrt{\pi}}\,\frac{\gamma_2}{\gamma_1}, \quad B_{\mathrm{melt}} = \frac{C_1 (T_f - T_0)}{L\sqrt{\pi}}\,\frac{\sqrt{a_1}}{\sqrt{a_2}}. \tag{8}$$

Correspondingly, we put Eqs. (4) and (5) into the form

$$\frac{A_{\text{fr}}\exp(-x_1^2)}{\operatorname{erf}(x_1)}-\frac{B_{\text{fr}}\exp(-x_2^2)}{\operatorname{erfc}(x_2)}=x_1, \tag{4'}$$

$$\frac{A_{\text{melt}}\exp(-x_2^2)}{\operatorname{erf}(x_2)}-\frac{B_{\text{melt}}\exp(-x_1^2)}{\operatorname{erfc}(x_1)}=x_2. \tag{5'}$$

Since the variables x_1 and x_2 are related by the relation

$$x_2=x_1\sqrt{a_1/a_2}, \tag{9}$$

Eqs. (4') and (5') also have a single positive root as x_1 and x_2 vary from zero to $+\infty$.

In order to determine the values of the root sought, we used a modified method of successive approximations and in so doing the zeroth approximation was determined graphically (Fig. 1).

The principle of the modification used was first formulated by Vegstein and is described in [7] for functions with a positive derivative. We used this modification for functions with a negative derivative. Very good results are obtained under the condition that the zeroth approximation is chosen so that the successive values of functions described by the left side of Eqs. (4') or (5') do not go beyond the first quadrant.

Using the modification indicated we developed an algorithm for solving Eqs. (4') or (5'). In the absence of the appropriate mathematical algorithms, the main difficulty in finding the roots of (4') and (5') lies in determining the values of Gauss' error function

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt. \tag{10}$$

We had to formulate a special program to calculate the integral (10). In doing so we used the algorithm presented in [7]. The values of the integral in the segment $0\leqslant x\leqslant 2.9$ were calculated using an expansion of the integrand in a Taylor series

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\left(x-\frac{x^3}{1!3}+\frac{x^5}{2!5}-\frac{x^7}{3!7}+\ldots\right). \tag{11}$$

In this case, for $0\leqslant x\leqslant 1.2$ we considered terms up to the term containing x^{13} in Eq. (11). For $1.2<x\leqslant 2.9$, the terms of the series in (11) were calculated until the term in the series leading to the required accuracy of the calculations was reached. For values $x>2.9$, we used the asymptotic series

$$\operatorname{erf}(x)=1-\frac{\exp(-x^2)}{x\sqrt{\pi}}\left(1-\frac{1}{2x^2}+\frac{1\cdot 3}{(2x^2)^2}-\frac{1\cdot 3\cdot 5}{(2x^2)^3}+\frac{1\cdot 3\cdot 5\cdot 7}{(2x^2)^4}\right). \tag{12}$$

All values of the integral were calculated with an accuracy of four decimal places. The results obtained using this algorithm were compared with the corresponding values of erf(x) presented in [8]. With some exceptions (disagreement in the fourth decimal place), the results of the calculation agreed.

This program for calculating erf(x) is a subroutine for the program that we developed for organizing the initial data [9], the algorithm for which reduces to the following.

1. Using a special subroutine, the root of Eqs. (4') or (5') is calculated and, the corresponding value of the coefficient β is calculated from Eqs. (6).

2. The location of the PTF at time t_0, taken as the initial time for the calculation, is calculated using Eq. (3).

3. The temperature profile in the zone $0<z<\zeta(t)$ is calculated using Eq. (1) for the same time.

4. The temperature profile in the zone $z>\zeta(t)$, i.e., the zone in which the medium is located in the initial aggregate state, is calculated using Eq. (2). The calculation according to (2) terminates when the condition $|T_0-T_V(z,\ t_0)|\leqslant 10^{-4}$ is satisfied.

The program developed for calculating solutions of (1)-(3) with the help of a computer

TABLE 1. Values of β Satisfying (4) for Freezing of Water

T_w, °C	T_0, °C					
	0	5	10	15	20	30
—1	0,00697	0,00613	0,00542	0,00483	0,00433	0,00355
—2	0,00985	0,00894	0,00814	0,00745	0,00685	0,00585
—3	0,01205	0,01109	0,01023	0,00948	0,00882	0,00769
—4	0,01390	0,01289	0,01200	0,01120	0,01050	0,00927
—5	0,01552	0,01448	0,01355	0,01272	0,01196	0,01067
—6	0,01699	0,01591	0,01495	0,01409	0,01330	0,01194
—7	0,01833	0,01723	0,01624	0,01534	0,01453	0,01311
—8	0,01957	0,01845	0,01743	0,01654	0,01567	0,01420
—9	0,02074	0,01969	0,01855	0,01760	0,01674	0,01522
—10	0,02184	0,02066	0,01960	0,01863	0,01775	0,01619
—20	0,03058	0,02922	0,02799	0,02685	0,02580	0,02393
—30	0,03710	0,03561	0,03425	0,03299	0,03183	0,02975
—40	0,04245	0,04086	0,03940	0,03805	0,03680	0,03456
—50	0,04703	0,04536	0,04383	0,04240	0,04108	0,03870
—60	0,05108	0,04930	0,04773	0,04624	0,04486	0,04237
—70	0,05471	0,05291	0,05124	0,04970	0,04827	0,04567
—80	0,05801	0,05616	0,05445	0,05286	0,05137	0,04869
—90	0,06104	0,05914	0,05739	0,05576	0,05424	0,05147
—100	0,06384	0,06191	0,06012	0,05845	0,05690	0,05407

TABLE 2. Values of β Satisfying (5) for Melting of Ice

T_w, °C	T_0, °C					
	—0	—5	—10	—15	—20	—30
1	0,00359	0,00258	0,00191	0,00148	0,00119	0,00085
2	0,00507	0,00399	0,00319	0,00260	0,00217	0,00160
3	0,00620	0,00508	0,00421	0,00354	0,00302	0,00229
4	0,00714	0,00601	0,00509	0,00436	0,00377	0,00293
5	0,00796	0,00682	0,00587	0,00510	0,00446	0,00352
6	0,00870	0,00755	0,00658	0,00577	0,00510	0,00408
7	0,09381	0,00822	0,00723	0,00639	0,00569	0,00461
8	0,01001	0,00884	0,00783	0,00698	0,00625	0,00511
9	0,01059	0,00942	0,00840	0,00753	0,00678	0,00558
10	0,01114	0,00996	0,00893	0,00805	0,00728	0,00604
20	0,01544	0,01424	0,01316	0,01219	0,01132	0,00983
30	0,01855	0,01735	0,01626	0,01526	0,01435	0,01275
40	0,02104	0,01985	0,01876	0,01775	0,01682	0,01516
50	0,02313	0,02195	0,02086	0,01985	0,01891	0,01722
60	0,02493	0,02377	0,02269	0,02168	0,02073	0,01902
70	0,02653	0,02538	0,02430	0,02330	0,02235	0,02063
80	0,02795	0,02682	0,02575	0,02475	0,02381	0,02208
90	0,02924	0,02812	0,02706	0,02607	0,02513	0,02341
100	0,03042	0,02931	0,02826	0,02727	0,02634	0,02462

can be used to organize the initial data when analyzing numerically problems involving prediction of freezing [10, 11].

Since for a very short time interval the solution of the axisymmetrical problem is close to the solution of the corresponding two-dimensional problem, in order to obtain the "initial solution" in studying the dynamics of freezing (melting) of a medium in pipes or around them it is also possible to use the program developed for organizing the initial data. In particular, we used it in solving problems involving the congealing of petroleum products in a pipeline when pumping is stopped [9].

In its turn, knowning β, it is easy to realize the solution (1)-(3) without using a computer. In this connection, we calculated on the M-4030 computer the values of β for water and ice for different temperature conditions. The results obtained are presented in Tables 1 and 2.

NOTATION

T_0, initial temperature of the medium; T_f, temperature of the phase transition; t, time; ζ, location of the phase transition front (PTF); T_u, a_u, temperature and thermal diffusivity of the medium in a region situated between the surface z = 0 and the PTF; T_v, a_v, temperature and thermal diffusivity of the medium in the region $z > \zeta(t)$; λ, C, γ, thermal conductivity, specific heat capacity, and density of the medium, respectively; L, specific heat of the phase transition.

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